MSE 203 Continuum Mechanics: Examples and Questions

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Here a series of additional examples and questions are set; we will select some to do in lectures, some for tutorials, and some will be left for private study. Where marks are indicated, this relates to the test marking scheme; the test will be composed of 6 questions are will be two hours in duration. Exam questions will require about twice as much effort as the test questions, and will be similar in level.

Example 1

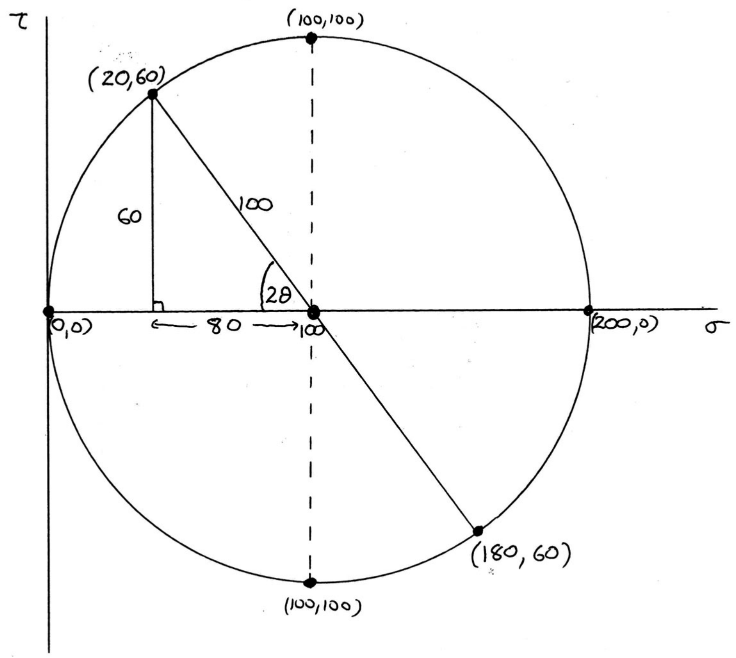
A material is subjected to the following state of stress

$$\sigma = \begin{pmatrix} 20 & 60 \\ 60 & 180 \end{pmatrix} \text{MPa} \tag{1}$$

Find the Principal Stresses, Maximum Shear Stress and the angle between these stress axes and the principal stress axes.

Solution 1

First we draw Mohr's Circle. The two stresses $(20, 60)$ and $(180, 60)$ are on opposite sides of the circle, as follows;



The Middle of the circle has to be at $20 + \frac{1}{2}(180 - 20) = 100\,\text{MPa}$.
The radius r we can calculate from the triangle in Mohr's circle as

$$r^2 = 60^2 + 80^2 \tag{2}$$

and so the radius is 100 MPa.

So the Principal Stresses are 0 and 200 MPa. The maximum shear stress is the radius of the circle, or 100 MPa.

Finally the angle in Mohr's circle is twice the angle in real space since Mohr's circle is defined in terms of 2θ, so we can write

$$\tan 2\theta = \frac{60}{80} \tag{3}$$

So $\theta = 18.43°$.

We can also solve this problem by finding the eigenvalues and eigenvectors, as follows;

$$\sigma = \begin{vmatrix} 20-\lambda & 60 \\ 60 & 180-\lambda \end{vmatrix} = 0 \tag{4}$$

So we can find the determinant;

$$(20-\lambda)(180-\lambda) - 60^2 = 0 \tag{5}$$

$$\lambda^2 - 200\lambda + 3600 - 3600 = 0 \tag{6}$$

$$\lambda(\lambda - 200) = 0 \tag{7}$$

So the solutions are that λ, and hence the principal stresses, are 0 and 200 MPa. The maximum shear stress is then given by half the difference between the maximum and minimum principal stresses, $\frac{1}{2}(\sigma_1 - \sigma_2) = 100$ MPa, c.f. Mohr's circle.

To find the angle between these axes we must find the eigenvectors, so we can write

$$\sigma = \begin{pmatrix} 20 & 60 \\ 60 & 180 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \tag{8}$$

We must solve this for each value of λ. For $\lambda = 0$ we can write

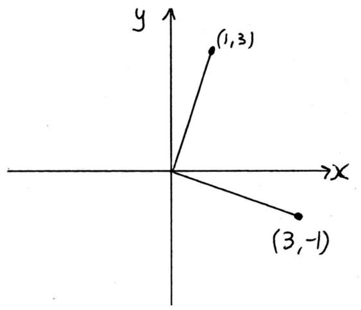
$$20x + 60y = 0 \tag{9}$$

and so $(x, y) = (3, -1)$ is a solution. For $\lambda = 200$ we can write

$$20x + 60y = 200x \tag{10}$$

$$60y = 180x \tag{11}$$

and so $(x, y) = (1, 3)$. We can show these vectors as;



We can find the angle between these eigenvectors and the axes using the dot product

$$\cos\theta = \frac{a.b}{|a||b|} \tag{12}$$

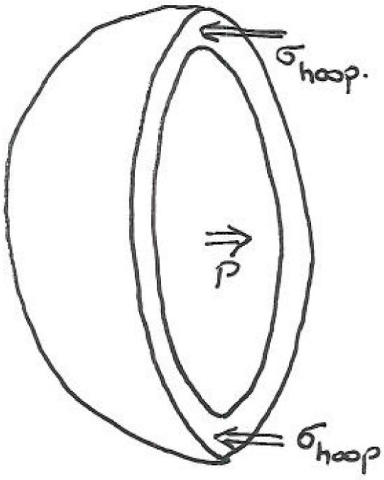
So the angle between $(3, -1)$ and the x-axis $(1, 0)$ is given by

$$\cos\theta = \frac{(3,-1).(1,0)}{|3,-1||1,0|} = \frac{3}{\sqrt{10}} \tag{13}$$

So this gives $\theta = 18.43°$, as before.

Question 1

A spherical pressure vessel in a Magnox power station of diameter 10m and wall thickness 75mm is internally pressurised with CO_2 gas at 20 bar (2MPa). By taking a section through the vessel (as with the thin pipe example in the notes, section 1.5), find the hoop stress in the material.



Whichever way you take the section, the hoop stresses will be the same. By inspection, we can deduce that the shear stress will be zero, and because the wall can be assumed to be thin, the radial stress will be zero. Therefore write down a stress matrix describing the state of stress in the vessel.

Question 2

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 0 & 25 \\ 25 & 0 \end{pmatrix} \text{MPa} \tag{14}$$

along its x and y axes.

Find the principal stresses and the maximum shear stress. What is the angle between the plane of maximum shear and the principal stresses?

Question 3

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 150 & 0 \\ 0 & 0 \end{pmatrix} \text{MPa} \tag{15}$$

along its x and y axes.

Find the principal stresses and the maximum shear stress. What is the angle between the plane of maximum shear and the principal stresses?

Question 4

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 150 & 25 \\ 25 & 0 \end{pmatrix} \text{MPa} \tag{16}$$

along its x and y axes.

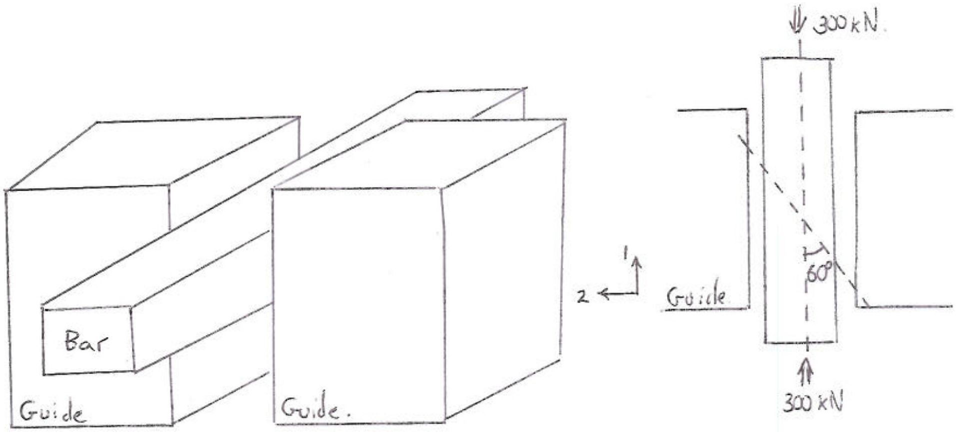
Find the principal stresses. At what angle to the x axis are these principal stresses found?

Question 5

A torsion test is performed on a round bar of radius 15 mm. A shear stress is applied, which is measured at the surface and found to be of magnitude 70 MPa. An additional compressive force of 50 kN is applied to the ends of the bar. Find the values of the principal stresses and the maximum shear stress. The example in section 2.1 of the notes may be useful.

Question 6

A steel bar 50mm × 50mm in section is, when unloaded, free to slide between (infinitely stiff) 100 mm long parallel guides spaced 50.005 mm apart. When the bar is subjected to an axial load, the bar expands due to Poisson's ratio in the transverse and normal directions (e.g. use the generalised Hooke's Law: $E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3)$). However, the bar is constrained to have a maximum width given by the spacing of the guides, and so if the axial load is high enough, the guides will impart a force on the bar to achieve this constraint.



Calculate the restraining force imparted by the guides when the axial load is 300 kN, given that the Young's modulus of the steel is 200 GPa and that its Poisson's ratio is 0.3. Therefore write down the stress matrix. Draw Mohr's circle of stress and therefore calculate the normal and shear stress on a plane inclined at 60° to the horizontal axis of the bar.

Question 7

A hollow shaft of outer diameter 75mm and wall thickness 5.5mm rotates at 5000 rpm and transfers a power of 1 MW. Using the equations $P = T\omega$, $T/J = \tau/R$ and knowing that $J = \frac{1}{2}\pi(R^4 - r^4)$, find the shear stress in the outer skin of the shaft. Using Mohr's circle, find the principal stresses.

[5 marks]

Question 8

A hydraulic oil pressure storage accumulator takes the form of a *thin* spherical pressure vessel of diameter 15cm and wall thickness 8.75mm. At its test pressure, it is internally pressurised with oil at 350 bar (35MPa). By drawing appropriate section(s) through the vessel and taking a force balance, find the stress state in the walls of the vessel and write down its stress matrix. Explain using Mohr's circle why the maximum shear stress must be 0.

[5 marks]

Question 9

A thin-walled cylindrical pressure vessel used as an air tank by divers with a volume of 12 L (0.012 m^3) contains air at a pressure of up to 240 bar (24 MPa). When empty, it is designed to be neutrally buoyant (neither sink nor float) in water of density 1000kgm^{-3}. The vessel is a right cylinder made of a low carbon steel of density 7050kgm^{-3} with an even thickness and has a diameter of 180mm. Find the thickness of the vessel.

Therefore deduce the stress state in the vessel, by drawing appropriate section(s) through the vessel and taking a force balance, and write down the stress matrix.

Finally draw Mohr's circle and therefore write down the stress matrix for the state of maximum shear.

[5 marks]

Question 10

A 'cruciform' materials test specimen is subjected to a longitudinal tensile stress of 200 MPa and a transverse compressive stress of 50 MPa. Write down the stress matrix and draw Mohr's circle.

Find the stress state at maximum shear. At what angle to the longitudinal stress axis is the plane of maximum shear found?

Finally, find the normal stress at an angle of 30° clockwise from the longitudinal stress axis.

[5 marks]

Question 11

A pipe is subjected to a hoop stress of 50 MPa, an axial stress of 60 MPa and a hoop-axial shear stress of 80 MPa. All the other stresses are zero. Write down the stress matrix and find the principal stresses and maximum shear stress. At what angle to the axis of the tube is the principal stress axis found? Finally, write down the hydrostatic stress.

[5 marks]

Question 12

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 300 & 200 & 0 \\ 200 & 0 & 100 \\ 0 & 100 & 300 \end{pmatrix} \text{MPa}$$

find the principal stresses, given that one of them is 300 MPa.

[5 marks]

Question 13

A new *bcc* Ti alloy is found to have $C_{11} = 125$, $C_{12} = 90$ and $C_{44} = 30$ GPa. It is loaded along the [100] direction in a tensile test such that

$$\begin{pmatrix} \sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 125 & 90 & 90 & 0 & 0 & 0 \\ 90 & 125 & 90 & 0 & 0 & 0 \\ 90 & 90 & 125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_4 \\ \varepsilon_4 \end{pmatrix} \tag{17}$$

Solve these simultaneous equations to find ε_1 and thus find the apparent stiffness of the crystal in the [100] direction, $E^{[100]}$.

[5 marks]

Question 14

A weld in a ferritic (*bcc*) water pipe is known to be in a state of plane stress. In the neutron diffractometer, the {211} plane is used to measure the change in lattice spacing in the two remaining principal axes in the pipe, which are the hoop and axial directions. The steel has a Young's modulus E of 220 GPa and Poisson's ratio ν of 0.3.

First, given that $d = a/\sqrt{h^2 + k^2 + l^2}$, $a = 2.86$ Å and that the wavelength of neutrons used is $\lambda = 1.54$ Å, find the diffraction angle of the {211} peak in the strain-free material.

This diffraction peak is found to move by −0.001° for the hoop direction and +0.0003° for the axial direction (remember that diffraction angles are given in 2θ). Given that $\varepsilon = \Delta d/d_0 = -\Delta\theta \cot\theta$, find the strain in each direction. Finally, apply the Generalied Hooke's Law, $E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3)$, to both these directions and therefore write down the stress state in the material.

[5 marks]

Question 15

A pipe is subjected to a hoop stress of 150 MPa, an axial stress of 30 MPa and a hoop-axial shear stress of 80 MPa. All the other stresses are zero. Write down the stress matrix and find the principal stresses and maximum shear stress. At what angle to the hoop axis of the tube is the maximum principal stress axis found? Finally, write down the hydrostatic stress $\sigma_H = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$.

[5 marks]

Question 16

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 300 & 200 & 0 \\ 200 & 100 & 100 \\ 0 & 100 & 300 \end{pmatrix} \text{MPa}$$

Find the principal stresses, given that one of them is 300 MPa.

[5 marks]

Question 17

A new *bcc* Ti alloy is found to have $C_{11} = 125$, $C_{12} = 90$ and $C_{44} = 30$ GPa. Find the Bulk Modulus of the alloy by applying a hydrostatic stress state to the single crystal, remembering that the hydrostatic stress $\sigma_H = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$, that the dilatation $\Delta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{22}$ and that the Bulk Modulus K is given by $\sigma_H = K\Delta$.

[5 marks]

Question 18

A weld in a ferritic (*bcc*) iron (steel) sheet is known to be in a state of plane stress. At an energy-dispersive synchrotron X-ray beamline, the diffraction angle 2θ is fixed at $5\,^{\circ}$. The change in the wavelength of the $\{211\}$ plane is used to measure the change in lattice spacing in the principal axes of the sheet, which are the longitudinal (welding) and transverse directions. The steel has a Young's modulus E of 220 GPa and Poisson's ratio ν of 0.3.

(i) Find the nominal diffraction wavelength for the $\{211\}$ peak in the strain-free material, given that $d = a/\sqrt{h^2 + k^2 + l^2}$ and $a \simeq 2.86$ Å.

(ii) This diffraction peak is found at 0.101895 Å in all directions in the strain free material far from the weld. In the peak stress region near the weld, it is found at 0.101944 Å for the longitudinal direction and 0.101872 Å for the transverse direction. Given that $\varepsilon = \Delta\lambda/\lambda_0 = \Delta d/d_0$, find the strain in each direction.

(iii) Using $\sigma_{11} = \frac{E}{1+\nu}\varepsilon_{11} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$, determine the stress state in the material.

[5 marks]

Question 19

The Tresca criterion states that $\sigma_y = \sigma_1 - \sigma_3$ and the Von Mises Criterion that $2\sigma_y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$.

Sketch the loci (line described) of both criteria on the plane where one of the principal stresses σ_2 is zero, and explain the shapes you have drawn. Identify the values of the stresses at $\sigma_1 = \pm\sigma_3$.

[5 marks]

Question 20

A camping gas cylinder (thin walled cylindrical pressure vessel) 10 cm in diameter is made with "thermoemchanically processed" sheet steel 0.3 mm thick with a yield stress of 500 MPa. It is recommended that it is only filled to a pressure 1/2 as great as that which would cause yielding in the walls. Using the Von Mises criteria, determine the maximum pressure to which the cylinder should be filled.

[5 marks]

Question 21

A hollow titanium shaft of outer diameter 30mm, wall thickness 2mm and with a yield stress of 1400 MPa is used in a wrench to apply a Torque to, *e.g.* vehicle wheel nuts. Using the equations $T/J = \tau/R$ and $J = \frac{1}{2}\pi(R^4 - r^4)$, derive an expression for the shear stress in the outer skin of the shaft. Using Mohr's circle, find the principal stresses. Therefore, if the shaft is to be used at only half is yield stress, find the maximum Torque that can safely be applied.

[5 marks]

Question 22

A cylindrical pipe of (inner) radius $r = 25\,\mathrm{mm}$ and thickness $t = 5\,\mathrm{mm}$ is pressurised with a gas at 300 bar (30 MPa). In addition, this shaft is subjected to a torque $T = 1.2\,\mathrm{kN\,m}$. Given that the shear stress at the outer radius R is given by $\tau = TR/J$, where the polar moment of inertia $J = \frac{1}{2}\pi(R^4 - r^4)$, find the shear stress in the bar. Further, assuming the pipe is thin, find the axial and hoop stresses and write down the complete stress matrix. Therefore find the three principal stresses in the pipe.

[5 marks]

Question 23

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 100 & 0 & -50 \\ 0 & 100 & 50 \\ -50 & 50 & 150 \end{pmatrix} \mathrm{MPa}$$

Given that the principal stresses are 100, 50 and 200 MPa, find a corresponding right-handed set of three unit eigenvectors corresponding to the principal axes. What are the angles these make with the original x, y and z axes?

[5 marks]

Question 24

An aluminium (*fcc*) component is known to be in a state of plane strain, such that $\varepsilon_{33} = 0$. The strains are measured in a neutron diffractometer using neutrons of wavelength 3.300 Å, using the {111} Al peak. In the 11 and 22 directions, the diffraction peak is found at 89.62 and 89.87°, respectively.

The strain-free lattice parameter of this material has been found to be 4.051 Å, the Young's modulus is 70 GPa and Poisson's ratio $\nu = 0.3$.

It is given that $\sigma_{11} = \frac{E}{1+\nu}\varepsilon_{11} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$, that $\varepsilon = -\partial\theta \cot\theta$, $\lambda = 2d \sin\theta$ and $d = a/\sqrt{h^2 + k^2 + l^2}$ (remember, the diffraction angle is 2θ).

Therefore determine the stress state in the material, assuming there are no shears.

[5 marks]

Question 25

For a cubic single crystal loaded along [100], find an expression for Poisson's ratio in the [010] and [001] directions, and therefore find an expression for Poisson's ratio in the [011] direction for loading along [100].

[5 marks]

Question 26

By regulation (FIFA, Law 2), a football is inflated to an overpressure of 1 atm (0.1 MPa), has a circumference of 70 cm and a weight of 450 g. Assuming it is a thin sphere made with leather of density $1400\,\mathrm{kg\,m^{-3}}$, find the stress state in the leather and compare this to its strength of $> 20\,\mathrm{MPa}$, using an appropriate yield criterion.

[5 marks]

Question 27

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 60 & 20 & 0 \\ 20 & 90 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{MPa}$$

Find the principal stresses and maximum shear stress in the material. Find the normal stress along an axis 30° from the axis of the maximum principal stress towards the smallest principal stress (remember, there are three principal stresses).

[5 marks]

Question 28

A new *bcc* Ti alloy is found to have $C_{11} = 125$, $C_{12} = 90$ and $C_{44} = 30\,\text{GPa}$. Find the stiffness in the [110] direction.

This can be achieved by applying a uniaxial stress state along the [110] / [1$\bar{1}$0] axes; then Mohr's circle can be used to find the stresses on the [100] / [010] axes, allowing the strains on these axes to be found; Mohr's circle can then be used again to rotate back to the original axis set. Before rotating back, remember that $\varepsilon_{12} = \frac{1}{2}\varepsilon_6$.

[5 marks]

Answers

Question 1

We simply balance the forces to find

$$\pi r^2 P = 2\pi r t \sigma_h$$

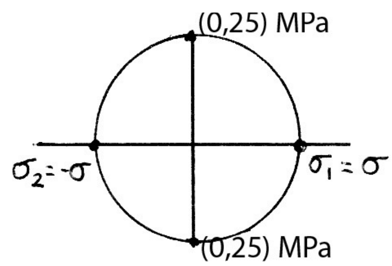
where t is the thickness and r the radius. Therefore using d as the diameter ($= 2r$),

$$\sigma_h = \frac{Pd}{4t} = \frac{2 \times 10}{4 \times 0.075} = 66.7\,\text{MPa}$$

This stress is applied in two directions (imagine taking sections through the vessel both horizontally and vertically). Therefore we have the stress matrix

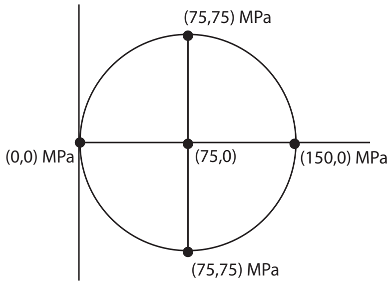
$$\sigma = \begin{pmatrix} 66.7 & 0 \\ 0 & 66.7 \end{pmatrix} \text{MPa}$$

Question 2



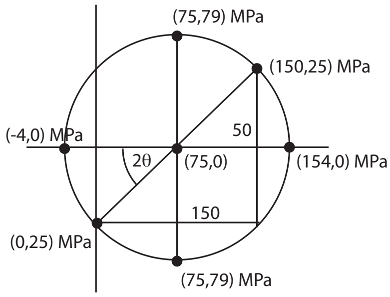
We can make a Mohr's circle for this situation as shown above, *i.e.* a circle, centred on the origin, of radius 25 MPa. The Principal stresses are therefore at the points of intersection of the circle with the horizontal axis, ±25 MPa. The angle between these points and the maximum shear points in Mohr's circle is 90°, and Mohr's circle is constructed in 2θ, so the angle between the principal stresses and the plane of maximum shear is 45°. This is a general result, for any stress state.

Question 3



The stress matrix describes a circle with the points (0,0) and (150,0) on opposite sides, shown above. Clearly, the centre of the circle is at (75,0) and the circle is of radius 75 MPa. So the principal stresses are those given, 150 and 0 MPa. The maximum shear stress is the radius, 75 MPa, and the stress state in this condition is $\sigma = \begin{pmatrix} 75 & 75 \\ 75 & 75 \end{pmatrix}$ MPa. The angle between the plane of maximum shear and the principal stresses is 45°, as in Question 2 and for the same reasons.

Question 4



Again, we start by drawing Mohr's circle with the stress state given on opposite diameters, *e.g.* the points $(150, 25)$ and $(0, 25)$. We can draw the triangle indicated, which has perpendicular lengths of 150 and $2 \times 25 = 50$. Therefore the long side of the triangle has a length of $\sqrt{150^2 + 50^2} = 50\sqrt{3^2 + 1^2} = 50\sqrt{10} = 158$. The radius of the circle is half this diameter, or 79. The centre of the circle is at the average of the two normal stresses, or the middle of the triangle, which is at $(75, 0)$. So the principal stresses are $75 \pm 79 = 154$ and -4 MPa.

The maximum shear stress is the radius of the circle, or 79 MPa, and the stress state on the plane of maximum shear is therefore $\sigma = \begin{pmatrix} 75 & 79 \\ 79 & 75 \end{pmatrix}$ MPa. Notice that at the plane of maximum shear the two normal stresses have to be equal.

The angle 2θ in Mohr's circle can be calculated by considering the triangle formed, so $\tan 2\theta = 25/75$, which gives θ, the angle between the plane of principal stress and the original axes, as 9.2° [NB: the angle we should really have drawn should be between the original x-axis at (150,25) and the maximum principal stress at (154,0), but clearly this is the same angle].

Question 5

First we need to work out the stress state. The compressive stress along the axis is given by the force over the area, so this stress is $\frac{-50.10^3}{\pi(15.10^{-3})^2} = -10^9.\frac{50}{\pi 15^2} = -70.7$ MPa. The stress matrix is then given by

$$\sigma = \begin{pmatrix} 0 & 70 \\ 70 & -70.7 \end{pmatrix} \text{MPa}$$

Notice that if the bar was subjected to a torque, we could calculate the shear stress using the procedure in section 2.1 of the notes.

We can then draw Mohr's circle as in the previous questions. The radius is equal to the maximum shear stress and is given by

$$\tau_{\max} = \frac{1}{2}\sqrt{(0 - (-70.7))^2 + (2 \times 70)^2} = 78.42 \text{ MPa}$$

The centre of the circle is at $\frac{1}{2}(0 - 70.7) = -35.35$ MPa, so the principal stresses are at $-35.35 \pm 78.42 = 43$ and -114 MPa.

Question 6

Let us denote the transverse direction of the bar the y direction and the axial direction x. The maximum strain in the y direction, ε_y, will be given by $\varepsilon_y = (l - l_0)/l_0 = (50.005 - 50)/50 = 10^{-4}$. The stress in the x direction, σ_x, is given by the force over the area, so

$$\sigma_x = \frac{-300 \times 10^3}{(50 \times 10^{-3})^2} = -120\,\text{MPa}$$

No force is applied in the z-direction, so the stress must be zero. Applying the generalised Hooke's law in the x and y directions gives

$$\begin{aligned} \text{x}: \qquad & 200\,\text{GPa}\ \varepsilon_x = -120\,\text{MPa} - 0.3\sigma_y \\ \text{y}: \qquad & 200\,\text{GPa}(10^{-4}) = \sigma_y - 0.3(-120\,\text{MPa}) \end{aligned}$$

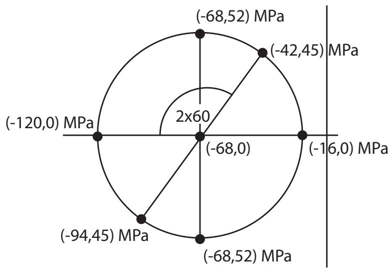
so we have two equations and two unknowns. We can rearrange the bottom equation to find that

$$\sigma_y = 200\,\text{GPa}(10^{-4}) + 0.3(-120\,\text{MPa}) = 20\,\text{MPa} - 36\,\text{MPa} = -16\,\text{MPa}$$

For completeness, we can then determine that the axial strain must be $\varepsilon_x = (-120 + 16 \times 0.3)/(200 \times 10^3) = -0.0576\%$.

The question asked us to find the *force* applied by the guides. They are 100mm long and the height of bar is 50mm, the force imparted by each guide must by $16 \times 100 \times 50 = 80$ kN.

We can therefore write down the stress matrix as $\sigma = \begin{pmatrix} -120 & 0 \\ 0 & -16 \end{pmatrix}$ MPa. These two stresses, -120 and -16 MPa, are the principal stresses. Therefore we can now draw Mohr's circle in the normal way;



The diameter of Mohr's circle is $-120 - (-16) = 104$ so the radius of the circle, and the maximum shear stress, is 52 MPa. The centre of the circle is at $-52 - 16 = -68$ MPa. We want the stress state at an angle of 60 degrees to the applied stress axis, so we rotate $2 \times 60 = 120°$ from that axis, as shown. We then find that the vertical (shear) side of the triangle has length $52 \sin 60 = 45.0$ MPa, and the horizontal side has length $52 \cos 60 = 26$ MPa. Therefore the normal stress at this location is $-68 + 26 = -42$ MPa and the shear stress is 45.0 MPa (and, by extension, the transverse stress from this axis we can work out to be $-68 - 26 = -94$ MPa).

Notice that if we used Equation 27 in the notes, $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2)$, we have to denote σ_1 as the larger (more positive) of the two stresses and σ_1 as the smaller in order for the equations to work out (otherwise the radius of the circle would be negative, which would be silly). Therefore the maximum shear stress is always a positive number, by convention.

If we have a stress problem with a negative shear stress, you will notice that in Equations 21 and 24 in the notes, τ_{xy} only appears as a squared term. Therefore it is safe to ignore the minus sign in a shear stress when plotting Mohr's circle. However, doing this creates a confusion in whether any rotation angles are positive or negative.

Question 7

First we can find that

$$T = P/\omega = \frac{10^6}{5000\frac{2\pi}{60}} = 1910\,\text{Nm}$$

Then we can find J;

$$J = \frac{1}{2}\pi\left([37.5 \times 10^{-3}]^4 - [32 \times 10^{-3}]^4\right) = 1.46 \times 10^{-6}\,\text{m}^4$$

[1 mark for T and J]

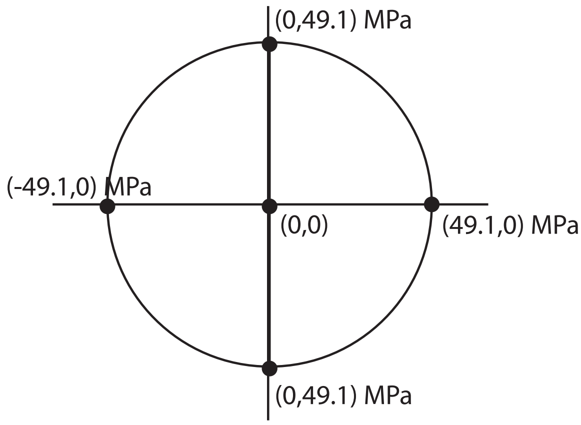
Therefore we can find the shear stress τ;

$$\tau = RT/J = 37.5 \times 10^{-3} \times 1910/1.46 \times 10^{-6} = 49.1\,\text{MPa}$$

[1 mark for τ]

We can immediately write the stress matrix as $\sigma = \begin{pmatrix} 0 & 49.1 \\ 49.1 & 0 \end{pmatrix}$ MPa and draw Mohr's circle;

[1 mark for the stress matrix]

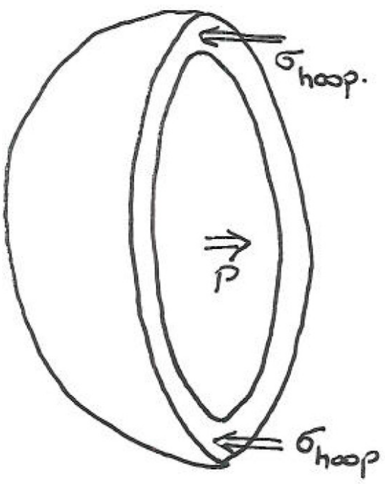


[1 mark for Mohr's circle]

Therefore the prinicpal stresses are $\pm 49.1\,\text{MPa}$.

[1 mark]

Question 8



We simply balance the forces to find

$$\pi r^2 P = 2\pi r t \sigma_h$$

[1 mark]

where t is the thickness and r the radius. Therefore

$$\sigma_h = \frac{Pr}{2t} = \frac{35 \times 0.075}{2 \times 0.00875} = 150\,\text{MPa}$$

[1 mark]

This stress is applied in two directions (imagine taking sections through the vessel both horizontally and vertically). Therefore we have the stress matrix

$$\sigma = \begin{pmatrix} 150 & 0 \\ 0 & 150 \end{pmatrix} \text{MPa}$$

[1 mark]

If we try to draw this on Mohr's circle, both points are at $(150, 0)$, *i.e.* Mohr's circle has zero radius. Therefore when the normal stresses are equal to each other, the maximum shear stress is zero, and the stress state is the same in all directions - it is isotropic.

[2 marks for realizing that Mohr's circle is a point]

Question 9

The volume V of a tank of radius r and height h will be $V = \pi r^2 h$, so the height is

$$h = \frac{V}{\pi r^2} = \frac{0.012}{\pi 0.09^2} = 0.47\,\text{m}$$

The volume of metal in the cylinder is then given by $2\pi rht + 2\pi r^2 t$ so to find the thickness we balance the mass of the tank against the mass of water displaced

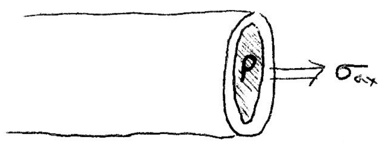
$$\rho_{\text{steel}}(2\pi rht + 2\pi r^2 t) = \rho_{\text{water}} V$$

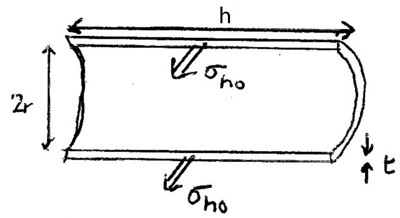
This leads us to find

$$t = \frac{\rho_{\text{water}}}{\rho_{\text{steel}}} \frac{0.012}{2\pi(rh + r^2)} = \frac{1}{7.05} \frac{0.012}{0.3167} = 5.375\,\text{mm}$$

[2 marks]

As in the example in the notes, we can take sections to find the axial and hoop stresses;





This leads us to find

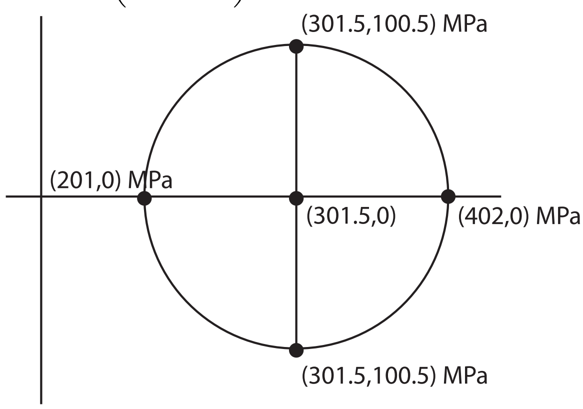
$$P\pi r^2 = 2\pi r t \sigma_{ax} \qquad 2rPh = 2ht\sigma_{ho}$$

so

$$\sigma_{ax} = \frac{Pr}{2t} = 201\,\text{MPa} \qquad \sigma_{ho} = \frac{Pr}{t} = 402\,\text{MPa}$$

[1 mark]

Therefore the stress matrix is $\sigma = \begin{pmatrix} 402 & 0 \\ 0 & 201 \end{pmatrix}$ MPa and Mohr's circle can be drawn



[1 mark]

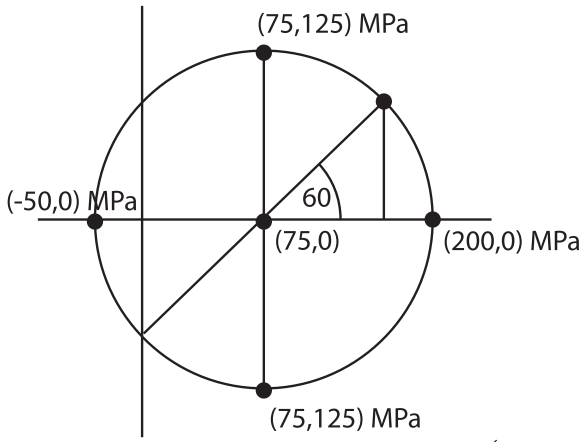
So we can write down the stress matrix for maximum shear: $\sigma = \begin{pmatrix} 301.5 & 100.5 \\ 100.5 & 301.5 \end{pmatrix}$ MPa.

[1 mark]

Question 10

The stress matrix is $\sigma = \begin{pmatrix} 200 & 0 \\ 0 & -50 \end{pmatrix}$ MPa and Mohr's circle can be drawn

[1 mark for the stress matrix]



So we can write down the stress matrix for maximum shear: $\sigma = \begin{pmatrix} 75 & 125 \\ 125 & 75 \end{pmatrix}$ MPa. These are at 90° to the principal axes in Mohr's circle, so they are at 45° to the principal axes in real space.

[2 marks for the radius, centre and matrix]

[1 mark for the angle]

We identify the triangle drawn, which is $30 \times 2 = 60°$ from the 200 MPa axis. The normal stress n on this triangle is the x-axis direction, so we can write

$$\cos 60 = n/125 \qquad \rightarrow \qquad n = 125 \cos 60 = 62.5\,\text{MPa}$$

[1 mark]

Question 11

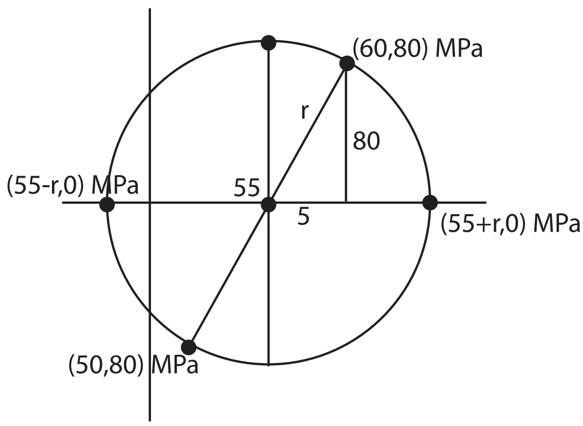
The stress matrix is

$$\begin{pmatrix} 60 & 80 \\ 80 & 50 \end{pmatrix} \text{MPa}$$

NB: by convention, $\sigma_{11} > \sigma_{22}$ (not enforced in mark scheme).

[1 mark for the stress marix]

Mohr's circle is therefore



Therefore the radius of Mohr's circle is $\sqrt{5^2 + 80^2} = 80.16\,\text{MPa}$, the maximum shear stress is equal to the radius and the prinicpal stresses are $55 \pm 80.16\,\text{MPa}$.

[2 marks]

The angle to the 60 MPa axial stress of the principal stress is given by $\tan 2\theta = 80/5$, so $\theta = 43.2°$.

[1 mark]

The hydrostatic stress invariant is the average of the normal stresses, so it is equal to $\frac{1}{3}(60 + 50 + 0) = 36.67\,\text{MPa}$ (because the third principal stress, the radial stress, is zero).

[1 mark]

Question 12

We need to find the eigenvalues of the stress matrix, solving $\det(M - \lambda I) = 0$. Hence we aim to solve

$$\sigma = 100 \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & -\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

Hence we find that, after cancelling the factor of 100 (which we have to remember to put back later),

$$\begin{aligned} (3-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & 3-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} &= 0 \\ (3-\lambda)\left[(-\lambda)(3-\lambda) - 1^2\right] - 2\left[2(3-\lambda)\right] &= 0 \\ (3-\lambda)(\lambda^2 - 3\lambda - 1) + (3-\lambda)(-4) &= 0 \\ (3-\lambda)(\lambda^2 - 3\lambda - 5) &= 0 \end{aligned}$$

Therefore the roots are given by $\lambda = 3, \frac{3}{2} \pm \frac{1}{2}\sqrt{3^2 + 4 \times 5} = 3, \frac{3}{2} \pm \frac{1}{2}\sqrt{29} = 3, 4.19, -1.19$, so the principal stresses are 300, 419 and -119 MPa.

[5 marks]

Question 13

The stress boundary condition describes the tensile test; there are no constraints on the strain. We merely need to solve the first two simultaneous equations, as follows

$$\begin{aligned} {}_1 &= C_{11}\varepsilon_1 + 2C_{12}\varepsilon_2 \\ 0 &= C_{12}\varepsilon_1 + (C_{11} + C_{12})\varepsilon_2 \end{aligned}$$

Therefore we can determine that $\varepsilon_2 = \frac{-C_{12}}{C_{11}+C_{12}}\varepsilon_1$ and therefore substitute back in to find that

$$E^{[100]} = \frac{\sigma_1}{\varepsilon_1} = C_{11} + \frac{-2C_{12}^2}{C_{11} + C_{12}} = 49.6\,\text{GPa}$$

[5 marks]

Question 14

Using Bragg's Law, $\lambda = 2d\sin\theta$, then since $d = a/\sqrt{6}$ then we can say that the diffraction angle (2θ) is given by

$$2\theta = 2\sin^{-1}\frac{\lambda}{2d} = 2\sin^{-1}\frac{1.54}{2 \times 2.86/\sqrt{6}} = 82.52°$$

[1 mark]

The strains are therefore given by

$$\begin{aligned} {}_h &= -(-0.001)\cot 41.26 = 0.00114 \\ \varepsilon_a &= -0.0003\cot 41.26 = -0.00034 \end{aligned}$$

[1 mark]

For each direction we can apply Hooke's law, $E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3)$ to find

$$\begin{aligned} \text{hoop}: \quad & E\varepsilon_h = \sigma_h - \nu(\sigma_a + 0) \\ \text{axial}: \quad & E\varepsilon_a = \sigma_a - \nu(\sigma_h + 0) \end{aligned}$$

where the stresses are now given in MPa. From the hoop equation, we find that $\sigma_h = \nu\sigma_a + E\varepsilon_h$. Substituting this back into the axial equation, we find that

$$E\varepsilon_a = \sigma_a - \nu(\nu\sigma_a + E\varepsilon_h)$$

$$\sigma_a = \frac{E(\varepsilon_a + \nu\varepsilon_h)}{1 - \nu^2}$$

[1 mark]

Which gives the results that $\sigma_a = 0$ and $\sigma_h = 251\,\text{MPa}$.

[1 mark]

Writing these as a stress matrix gives

$$\sigma = \begin{pmatrix} 251 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{MPa}$$

[1 mark]

Question 15

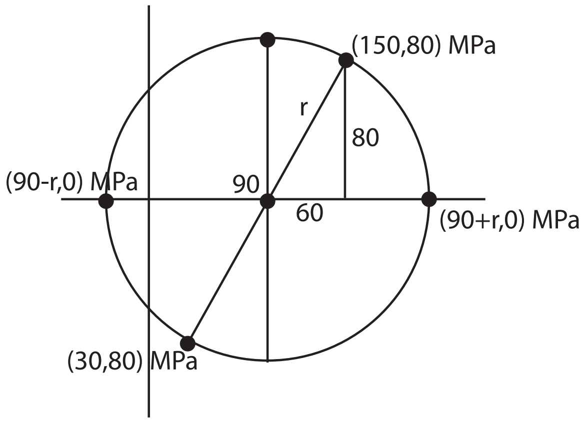
The stress matrix is

$$\begin{pmatrix} 150 & 80 \\ 80 & 30 \end{pmatrix} \text{MPa}$$

NB: by convention, $\sigma_{11} > \sigma_{22}$ (not enforced in mark scheme).

[1 mark for the stress marix]

Mohr's circle is therefore



Therefore the radius of Mohr's circle is $\sqrt{60^2 + 80^2} = 100\,\text{MPa}$ (3-4-5 triangle), the maximum shear stress is equal to the radius and the prinicpal stresses are $90 \pm 100\,\text{MPa}$.

[2 marks]

The angle to the 150 MPa hoop stress from the max principal stress is given by $\tan 2\theta = 80/60$, so $\theta = 26.6°$.

[1 mark]

The hydrostatic stress invariant is the average of the normal stresses, so it is equal to $\frac{1}{3}(150 + 30 + 0) = 60\,\text{MPa}$ (because the third principal stress, the radial stress, is zero).

[1 mark]

Question 16

We need to find the eigenvalues of the stress matrix, solving $\det(M - \lambda I) = 0$. Hence we aim to solve

$$\sigma = 100 \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

Hence we find that, after cancelling the factor of 100 (which we have to remember to put back later),

$$(3-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & 3-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 1-\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(1-\lambda)(3-\lambda) - 1^2\right] - 2\left[2(3-\lambda)\right] = 0$$

$$(3-\lambda)(\lambda^2 - 4\lambda + 2) + (3-\lambda)(-4) = 0$$

$$(3-\lambda)(\lambda^2 - 4\lambda - 2) = 0$$

Therefore the roots are given by $\lambda = 3, \frac{4}{2} \pm \frac{1}{2}\sqrt{(-4)^2 - 4 \times (-2)} = 3, 2 \pm \sqrt{6} = 4.45, 3, -0.45$, so the principal stresses are 445, 300 and -45 MPa.

[5 marks]

Question 17

We can describe a hydrostatic test by

$$\begin{pmatrix} \sigma_H \\ \sigma_H \\ \sigma_H \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 125 & 90 & 90 & 0 & 0 & 0 \\ 90 & 125 & 90 & 0 & 0 & 0 \\ 90 & 90 & 125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

Therefore we can write from the first three rows;

$$\sigma_H = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{12}\varepsilon_{33}$$
$$\sigma_H = C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{12}\varepsilon_{33}$$
$$\sigma_H = C_{12}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{11}\varepsilon_{33}$$

Adding up these we obtain $3\sigma_H = (C_{11} + 2C_{12})\Delta$, so by comparison

$$K = \frac{C_{11} + 2C_{12}}{3} = \frac{305}{3} = 101.7\,\text{GPa}$$

[5 marks]

Question 18

(i) $d = 2.86/\sqrt{6} = 1.16759$ Å. Using the approximation that $\sin\theta \simeq \theta$ for small angles, then $\lambda = 2 \times 1.16759 \times \frac{5}{2}\frac{\pi}{180} = 0.10189$Å.

[1 marks]

(ii) We can find the strains from;

$$\varepsilon_l = \frac{\delta\lambda}{\lambda} = \frac{49 \times 10^{-6}}{0.101895} = 481 \times 10^{-6}$$

$$\varepsilon_t = \frac{\delta\lambda}{\lambda} = \frac{-23 \times 10^{-6}}{0.101895} = -226 \times 10^{-6}$$

[1 marks]

(iii) $\frac{E}{1+\nu} = 169,230\,\text{MPa} = A$ and $\frac{\nu E}{(1+\nu)(1-2\nu)} = 126,920\,\text{MPa} = B$. We have three equations, and three unknowns;

$$\sigma_l = A\varepsilon_l + B(\varepsilon_l + \varepsilon_t + \varepsilon_z)$$

$$\sigma_t = A\varepsilon_l + B(\varepsilon_l + \varepsilon_t + \varepsilon_z)$$

$$0 = A\varepsilon_z + B(\varepsilon_l + \varepsilon_t + \varepsilon_z)$$

since the weld is in a state of plane stress. Therefore we can rearrange the third equation to find

$$-(A+B)\varepsilon_z = B(\varepsilon_l + \varepsilon_t)$$

so

$$\varepsilon_z = -(\varepsilon_l + \varepsilon_t)\frac{B}{A+B} = -109 \times 10^{-6}$$

and $\Delta = (\varepsilon_l + \varepsilon_t + \varepsilon_z) = 146 \times 10^{-6}$. So then we can substitute back to find $\sigma_l = 81.4 + 18.5 = 99.9\,\text{MPa}$ and $\sigma_t = -38.3 + 18.5 = -19.8\,\text{MPa}$. So the stress matrix is

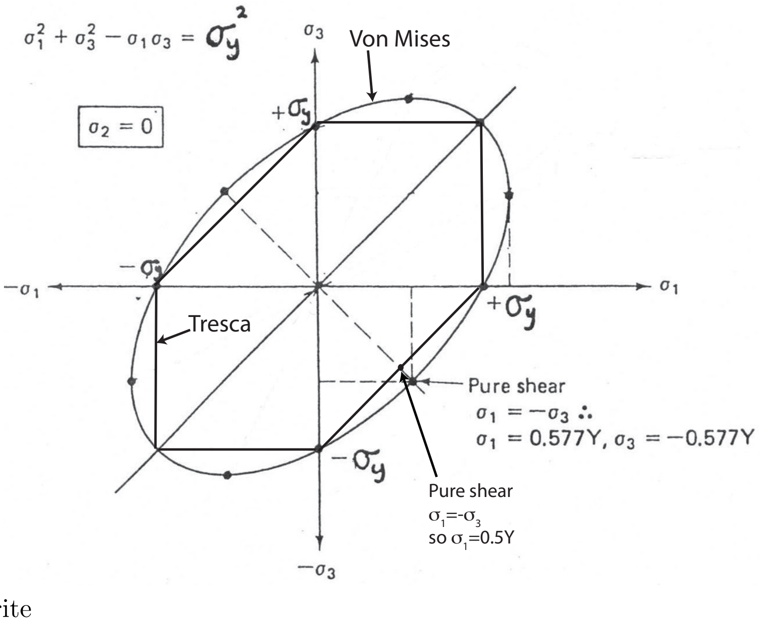
$$\begin{pmatrix} 100 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{MPa}$$

[3 marks]

Question 19

For Tresca, we identify each of the possible scenarios;

$\sigma_1 > \sigma_3 > 0$: The smallest principal stress is zero, so the line is $\sigma_y = \sigma_1 - 0$.
$\sigma_1 > 0 > \sigma_3$: σ_3 is negative, so the line is $\sigma_3 = \sigma_1 - \sigma_y$.
$\sigma_3 > \sigma_1 > 0$: The smallest principal stress is zero, so the line is $\sigma_y = \sigma_3 - 0$.
$0 > \sigma_3 > \sigma_1$: both are negative, σ_1 is more negative. 0 is the biggest stress.
$0 > \sigma_1 > \sigma_3$: same again.
$\sigma_3 > 0 > \sigma_1$: the stresses take opposite signs again.
This gives six sides to the hexagon, as drawn.
When $\sigma_1 = \sigma_3$ and $\sigma_2 = 0$, both equal the uniaxial yield stress. When $\sigma_1 = -\sigma_3$, $\sigma_{1,3} = \pm\frac{1}{2}\sigma_y$.



For Von Mises, we can write

$$2\sigma_y^2 = \sigma_1^2 + \sigma_3^2 + (\sigma_3 - \sigma_1)^2$$

$$\sigma_y^2 = \sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3$$

This is the equation of an ellipse, centred on the axes, as shown. Along the major axis of the ellipse, $\sigma_1 = \sigma_3$, then substitution will show that $\sigma_1 = \sigma_y$. Along the minor axis of the ellipse, $\sigma_1 = -\sigma_3$, then similarly $\sigma_1 = \frac{1}{\sqrt{3}}\sigma_y = 0.577\sigma_y$.

[5 marks]

Question 20

By drawing sections and performing a force balance, we can determine;

$$\sigma_l 2\pi r t = P\pi r^2 \rightarrow \sigma_l = Pr/2t$$

$$\sigma_h 2lt = Pl2r \rightarrow \sigma_h = Pr/t$$

Substituting these into Von Mises, we find that at yield

$$2\sigma_y^2 = \left(\frac{r}{2t}\right)^2 \left(P^2 + P^2 + (2P)^2\right)$$

so at the yielding pressure

$$P_y = \frac{2t\sigma_y}{\sqrt{3}r}$$

So the maximum pressure recommended is simply half this value;

$$P_y = \frac{t\sigma_y}{\sqrt{3}r} = \frac{300 \times 500}{\sqrt{3} \times 0.05} = 17.3\,\text{bar}$$

[5 marks]

Question 21

Rearranging the equations given, we find

$$\tau = \frac{RT}{J} = \frac{2RT}{\pi(R^4 - r^4)}$$

The principal stresses in Mohr's circle are then, trivially, $\pm\tau$ for the case of pure shear.

Substituting into Tresca, the third principal stress is zero and so

$$\sigma_y = \tau - -\tau = 2\tau$$

Substituting for τ and rearranging, we find that at yield

$$T = \frac{\sigma_y \pi (R^4 - r^4)}{4R}$$

Therefore if we are limited to half the yield stress, we find

$$T = \frac{700 \times 10^6 \pi (0.015^4 - 0.013^4)}{4 \times 0.015} = 810\,\text{Nm}$$

[5 marks]

Question 22

A cylindrical pipe of (inner) radius $r = 25\,\text{mm}$ and thickness $t = 5\,\text{mm}$ is pressurised with a gas at 300 bar (30 MPa). In addition, this shaft is subjected to a torque $T = 1.2\,\text{kN m}$. Given that the shear stress at the outer radius R is given by $\tau = TR/J$, where the polar moment of inertia $J = \frac{1}{2}\pi(R^4 - r^4)$, find the shear stress in the bar. Further, assuming the pipe is thin, find the axial and hoop stresses and write down the complete stress matrix. Therefore find the three principal stresses in the pipe.

[5 marks]

First, J is given by $J = \frac{\pi}{2}(0.03^4 - 0.025^4) = 6.588 \times 10^{-7}\ \text{m}^4$. Then the shear stress τ in the outer fiber is $\tau = 1200 \times 0.03/J = 54.6\,\text{MPa}$.

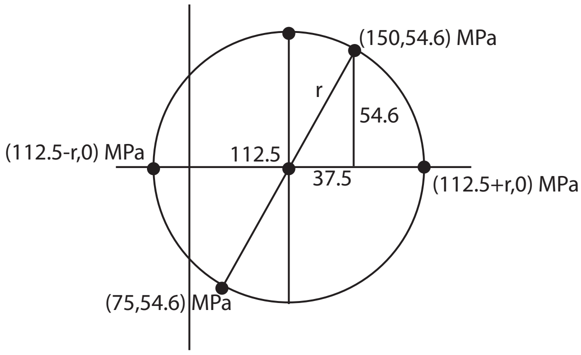
For a pipe, performing a stress balance in each direction gives $\sigma_l = Pr/2t$ and $\sigma_h = Pr/t$. So $\sigma_l = 30 \times 25/(5 \times 2) = 75\,\text{MPa}$ and $\sigma_h = 150\,\text{MPa}$.

(it is also possible to use other values for r in these formulae, e.g. taking the average radius to be 27.5 mm).

So the stress matrix is

$$\begin{pmatrix} 150 & 54.6 \\ 54.6 & 75 \end{pmatrix} \text{MPa}$$

Mohr's circle is therefore



So the radius is 66.2 MPa and therefore the principal stresses are 178.7, 46.3 and 0 MPa.

Question 23

A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 100 & 0 & -50 \\ 0 & 100 & 50 \\ -50 & 50 & 150 \end{pmatrix} \text{MPa}$$

Given that the principal stresses are 100, 50 and 200 MPa, find a corresponding right-handed set of three unit eigenvectors corresponding to the principal axes. What are the angles these make with the original x, y and z axes?

[5 marks]

We need to solve for the eigenvectors, *e.g.* solve simultaneous equations like $\sigma a = \lambda a$, where $a = (x, y, z)$ is an eigenvector.

For example, for the first eigenvalue of 100, we obtain the following three equations;

$$\begin{aligned} 100x + 0y - 50z &= 100x \\ 0x + 100y + 50z &= 100y \\ -50x + 50y + 150z &= 100z \end{aligned}$$

From the first and second lines, $z = 0$. So, from the third line, $x = y$, so $1/\sqrt{2}(1, 1, 0)$ is a valid unit eigenvector.

Similarly, other valid eigenvectors are $1/\sqrt{3}(1, -1, 1)$ for the eigenvalue of 50, and $1/\sqrt{6}(1, -1, -2)$ for the eigenvalue of 200 (or permutations thereof).

These make the following angles with the axes (x, y, z, in order);
$1/\sqrt{2}(1, 1, 0)$: 45, 45 and 90°.
$1/\sqrt{3}(1, -1, 1)$: 54.7, 125.3 and 54.7°.
$1/\sqrt{6}(1, -1, -2)$: 65.9, 114.1 and 144.7°.

Question 24

An aluminium (*fcc*) component is known to be in a state of plane strain, such that $\varepsilon_{33} = 0$. The strains are measured in a neutron diffractometer using neutrons of wavelength 3.300 Å, using the {111} Al peak. In the 11 and 22 directions, the diffraction peak is found at 89.62 and 89.87°, respectively.

The strain-free lattice parameter of this material has been found to be 4.051 Å, the Young's modulus is 70 GPa and Poisson's ratio $\nu = 0.3$.

It is given that $\sigma_{11} = \frac{E}{1+\nu}\varepsilon_{11} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$, that $\varepsilon = -\partial\theta \cot\theta$, $\lambda = 2d \sin\theta$ and $d = a/\sqrt{h^2 + k^2 + l^2}$ (remember, the diffraction angle is 2θ).

Therefore determine the stress state in the material, assuming there are no shears.

[5 marks]

First, $d = a/\sqrt{3} = 2.339$ Å. Then, $2\theta_0 = 2\sin^{-1}(\lambda/2d) = 89.74\,°$. So $\cot\theta \sim 1$ and

$$\varepsilon_{11} = -\delta\theta = -\frac{1}{2}(89.62 - 89.736) \times \pi/180 = 1.0 \times 10^{-3}$$

Similarly, $\varepsilon_{22} = -1.2 \times 10^{-3}$. So $\Delta = \varepsilon_{ii} = -0.2 \times 10^{-3}$.

So, given that $E/(1+\nu) = 53.85\,\text{GPa}$ and Lamé's constant is $E\{1+\nu)(1-2\nu) = 40.38\,\text{GPa}$, then

$$\sigma_{11} = 53.85 \times 1.0 - 0.2 \times 40.38 = 45.8\,\text{MPa}$$

Similarly, $\sigma_{22} = -72.7\,\text{MPa}$ and $\sigma_{33} = -8.1\,\text{MPa}$.

Question 25

For a cubic single crystal loaded along [100], find an expression for Poisson's ratio in the [010] and [001] directions, and therefore find an expression for Poisson's ratio in the [011] direction for loading along [100].

[5 marks]

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{pmatrix} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix}$$

Loading along the [100], we therefore find

$$\sigma = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{12}\varepsilon_3$$
$$0 = C_{12}\varepsilon_1 + C_{11}\varepsilon_2 + C_{12}\varepsilon_3$$

By symmetry, $\varepsilon_3 = \varepsilon_2$ (or, by setting up the third simultaneous equation and solving). Therefore,

$$\varepsilon_2 = -\varepsilon_1 \frac{C_{12}}{C_{11} + C_{12}}$$

Therefore, by comparing with the definition of Poisson's ratio $\nu = -\varepsilon_2/\varepsilon_1$ (from e.g. the generalised Hooke's law), we find that $\nu = \frac{C_{12}}{C_{11}+C_{12}}$.

[For the crystal in Q7, this gives $\nu = 0.42$].

For the [011] direction, we must rotate in the [010]-[001] plane. We are rotating a strain matrix of $\begin{pmatrix}\nu\varepsilon_1 & 0\\ 0 & \nu\varepsilon_1\end{pmatrix}$, *i.e.* one that is a pure dilatation, or a Mohr's circle with zero radius. So however we rotate in the plane, we get the same strains. So the Poisson's ratio in the [011] is also $\nu = \frac{C_{12}}{C_{11}+C_{12}}$.

Question 26

By regulation (FIFA, Law 2), a football is inflated to an overpressure of 1 atm (0.1 MPa), has a circumference of 70 cm and a weight of 450 g. Assuming it is a thin sphere made with leather of density $1400\,\text{kg}\,\text{m}^{-3}$, find the stress state in the leather and compare this to its strength of $> 20\,\text{MPa}$, using an appropriate yield criterion.

[5 marks]

The radius of the sphere must be $0.7/2\pi = 0.111\,\text{m}$. Therefore its area is $A = 4\pi r^2 = 0.156\,\text{m}^2$. The volume of leather contained must be given by $V = m/\rho = 0.45/1400 = 3.21 \times 10^{-4}\,\text{m}^3$. So the thickness of the ball is $V/A = t = 2.06\,\text{mm}$. The ball is a sphere, so performing a stress balance $\sigma_h 2\pi r = P\pi r^2$, so $\sigma_h = Pr/2t = 0.1 \times 0.111/(2\times 0.00206) = 2.70\,\text{MPa}$. Using the Von Mises criterion, then $2\sigma_{\text{VM}}^2 = (\sigma_h-\sigma_h)^2 + (\sigma_h-0)^2 + (0-\sigma_h)^2 = 2\sigma_h^2$, so $\sigma_{\text{VM}} = \sigma_h = 2.70\,\text{MPa}$, which is about 14% of the material's yield stress (Tresca, using the max and min principal stresses as σ_h and 0, gives the same result in this case of biaxial tension).

Question 27

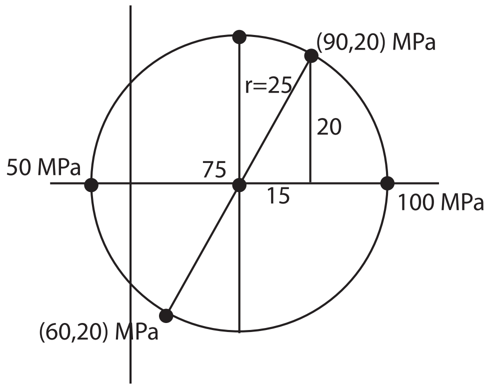
A material is subjected to a stress state of

$$\sigma = \begin{pmatrix} 60 & 20 & 0\\ 20 & 90 & 0\\ 0 & 0 & 0\end{pmatrix}\,\text{MPa}$$

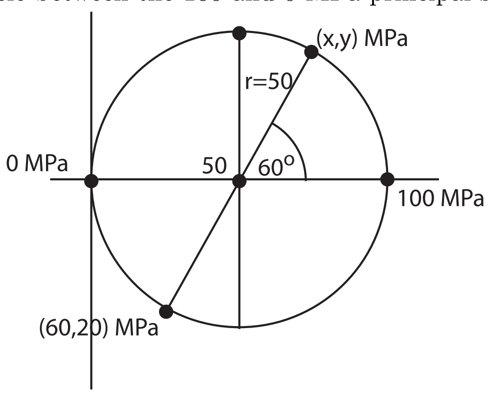
Find the principal stresses and maximum shear stress in the material. Find the normal stress along an axis 30° from the axis of the maximum principal stress towards the smallest principal stress (remember, there are three principal stresses).

[5 marks]

Mohr's circle is



The max shear stress is 25 MPa and the principal stresses are 100, 50 and 0 MPa.
Making a new Mohr's circle between the 100 and 0 MPa principal stresses and rotating 30°, we obtain



So the normal stress along the requested axis is $x = 50 + 50\cos 60 = 75\,\text{MPa}$.

Question 28

A new *bcc* Ti alloy is found to have $C_{11} = 125$, $C_{12} = 90$ and $C_{44} = 30\,\text{GPa}$. Find the stiffness in the [110] direction.

This can be achieved by applying a uniaxial stress state along the [110] / [$1\bar{1}0$] axes; then Mohr's circle can be used to find the stresses on the [100] / [010] axes, allowing the strains on these axes to be found; Mohr's circle can then be used again to rotate back to the original axis set. Before rotating back, remember that $\varepsilon_{12} = \frac{1}{2}\varepsilon_6$.

[5 marks]

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix}$$

For the stiffness in the [110] direction we apply a stress along [110] of $\begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix}$. We then need to rotate this by 45° to obtain the stress state on the $< x00 >$ axes. Using Mohr's circle, this will provide a stress state of $\begin{pmatrix}\sigma/2 & \sigma/2\\ \sigma/2 & \sigma/2\end{pmatrix}$ (rotating 90° in Mohr's circle).

We then apply this to our formula (the other stresses involving the 3 direction are 0 and of no concern);

$$\begin{aligned}
\sigma/2 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{12}\varepsilon_3 \\
\sigma/2 &= C_{12}\varepsilon_1 + C_{11}\varepsilon_2 + C_{12}\varepsilon_3 \\
0 &= C_{12}\varepsilon_1 + C_{12}\varepsilon_2 + C_{11}\varepsilon_3 \\
\sigma/2 &= C_{44}\varepsilon_6
\end{aligned}$$

So $\varepsilon_6 = \sigma/2C_{44} = \sigma/60$. So $\varepsilon_{12} = \sigma/120$. Clearly, $\varepsilon_1 = \varepsilon_2$ (subtract the first and second lines). Therefore $\varepsilon_3 = -2C_{12}\varepsilon_1/C_{11}$.

So

$$\sigma/2 = \varepsilon_1 \left(C_{11} + C_{12} + \frac{-2C_{12}^2}{C_{11}} \right) = \varepsilon_1(125 + 90 - 129.6)$$

So $\varepsilon_1 = \sigma/170.8$

So our strain matrix on the $<x00>$ axes is $\begin{pmatrix} \sigma/170.8 & \sigma/120 \\ \sigma/120 & \sigma/170.8 \end{pmatrix}$. We then rotate this back 45° to obtain the stress state on the [110] axes again, using Mohr's circle; trivially the radius is $\sigma/120$, so the (principal) strains are $\sigma/170.8 \pm \sigma/120$. Therefore the stiffness in the [100] direction, or σ/ε is

$$\frac{\sigma}{\sigma/170.8 + \sigma/120} = \frac{1}{1/170.8 + 1/120} = 70.5\,\text{GPa}$$